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Journal of Environmental Biology

ISSN: 0254-8704 (Print)
ISSN: 2394-0379 (Online)
CODEN: JEBIDP

Generalized population dynamic operator with delay based on fractional calculus

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Publication Info

Paper received:
04 November 2015

Revised received:
27 April 2016

Re-revised received:
05 May 2016

Accepted:
23 June 2016

Abstract

In population dynamics, a growing population consumes more food than a matured one that depends upon condition of individual species. This hints to neutral equations. In the present study, certain sufficient conditions for the existence of periodic solutions to a generalized Rayleigh-type equation with state dependent delay, based on fractional calculus concept was investigated.

Key words

Fractional calculus, Fractional differential operator, Fractional differential equation, Rayleigh equation, Periodic solution

Introduction

The intelligent movement of mathematics-biology interface, from mathematics to biology and from biology to mathematics, drives in both directions. Mathematical biology is building and will continue to create real influences and contributions. Contemporary mathematicians occasionally terminate physics as "applied mathematics" and thus, disregard the important characters that physics played in the growth of theoretical philosophies in fundamental mathematics. The planets formed the discipline of periodicity to systems. The Maxwell's equations and its alternatives drove the growth of partial differential equations. In the current era, crystallography which is partially accountable for the growth of group theory; quantum mechanics is a significant motivation for functional analysis, qualitative theory of differential equation, geometry and quantum field theory for string theory. The progress of novel measurement implements mathematics and statistics in biological problems. This not only have the traditional tools in ordinary and partial differential equations, but also rise in application in graph theory, algebraic statistic, fractional differential equation and random graphs, those have been employed to epidemiology, and gene networks (Durrett, 2008; Ibrahim *et al.,* 2016).

In population dynamics, normally a growing population consumes more food than a matured one that is restricted to individual species and this yields neutral equations. In 1904, Lord Rayleigh introduced a natural equation *i.e.,* $u''(t) + f(u'(t)) + \alpha u(t) = 0$, where f is a restoring controller of the size of population u, balancing with a constant $\alpha > 0$ to describe the population. The existence of periodic solutions of the Rayleigh differential equations has been studied by Gains and Mawhin (1977) and Deimling (1985). Later, various researchers modified the Rayleigh differential equations and established the existence of periodic solutions (Ren*et al*., 2011; Alzabut and Tunc, 2012; Xin and Zhao, 2015). Anguraj *et al*. (2015) investigated the outcome for the fractional neutral differential equations with random impulses. The outcomes were confirmed by applying Krasnoselskii's fixed point theorem. Abas *et al*. (2015) studied an existence attractive outcome for initial value problems of fractional order neutral differential equations with finite delay.

Fractional order differential equations have definitely proved in modeling of numerous diverse techniques and arrangements in physics, chemistry, biology, engineering, medicine and food processing (Ibrahim and Jahangiri, 2014; Ibrahim and Jahangiri, 2015).

In the present study, certain sufficient conditions for the existence of periodic outcomes to a fractional Rayleigh-type equation with state-dependent delay was investigated. Consequently, some generalized known works in the literature was generated and improved.

Materials and Methods

In the present, certain sufficient conditions were investigated for the existence of periodic outcomes to a fractional Rayleigh-type equation with state-dependent delay of the formula : $u''(t)+\beta(u(t))D^{\wp}u(t)+\phi(t,u'(t-\tau))+\psi[t,u(t-\iota(t,u(t)))]=\eta(t)$ (1)

where, $\tau \in [0, T]$, $t \geq \tau$, ϕ, $\psi : R^2 \rightarrow R$ are T-periodic continuous functions in $t \in [0, T]$, $T < \infty$, satisfying $\phi(t, 0) = \psi(t, 0) = 0$, $\iota : R^2 \rightarrow (0, \infty)$ is a positive continuous function, β: $R \rightarrow R$, η: $[0, T] \rightarrow R$ are T-periodic continuous functions in t, with $\int_0^T \eta(t)dt = 0$ and $D^{\wp}$ is the Riemann-Liouville fractional differential operator, which is defined by

$$D^{\wp}u(t) = \frac{d}{dt}\int_0^t \frac{(t-\tau)^{-\wp}}{\Gamma(1-\wp)}u(\tau)d\tau,\ t \in [0,T],\ \wp \in (0,1).$$

It is to be noted that Eq. (1) is a generalization of equations studied by Alzabut and Tunc (2012) and Xin and Zhao (2015).

Results and Discussion

In this section, the existence of periodic solutions was dealt. The Banach space is defined as $B := \{u \in C(R, R): u(t+T) = u(t), t \in J := [0, T]\}$, with maximum norm $\| u \| = \max_{t\in J} | u(t) |$. In the sequel, it was assumed $\overline{\beta} := \sup_{u\in B} | \beta(u(t)) |$. The following findings are as follows :

Theorem 3.1 : Suppose that there are positive constants λ_i, $i = 1, ..., 4$ such as

$$|\phi(t,u)| \leq \lambda_1 \| u \| + \lambda_2 < |\psi(t,u)| \leq \lambda_3 \| u \| + \lambda_4. \quad (2)$$

If $$\| u \| \leq \int_0^T |u''(s)|ds, \quad (3)$$

and

$$2T(\lambda_1+\lambda_3) + \frac{\overline{\beta}T^{1-\wp}}{\Gamma(1-\wp)} < 1.$$

Eq. 1 admits at least one T-periodic solution in B.

Proof : Since ϕ and ψ are T-Periodic functions in J, then we obtain

$$\int_0^T \{\phi(s,u'(s-\tau)) + \psi[s,u\left(s-\iota(s,u(s))\right)]\}ds = 0. \quad (4)$$

Consequently, there exists a $t_1 \in J$ such as

$$\phi(t_1,u'(t_1-\tau)) + \psi[t_1,u\left(t_1-\iota(t_1,u(t_1))\right)] = 0. \quad (5)$$

By the assumption (2), we conclude that

$$|\phi(t_1,u'(t_1-\tau))| = |\psi(t_1,u(t_1-\iota(t_1, u(t_1))))| \leq \lambda_1 \| u \| + \lambda_2.$$

But,

$$|\psi(t,u(t-\iota(t,u(t))))| > \lambda_1 \| u \| + \lambda_2,$$

Then

$$|u(t_1-\iota(t_1,u(t_1))| \leq \| u \|.$$

According to the above inequality, we define the following sets:

$$\Xi_1 := \{t: t \in J, u(t-\iota(t,u(t)) > \| u \|\}$$
$$\Xi_2 := \{t: t \in J, u(t-\iota(t,u(t)) < -\| u \|\}$$
$$\Xi_3 := \{t: t \in J, |u(t-\iota(t,u(t))| \leq \| u \|\}.$$

In view of Eq. 4 :

$$\int_{\Xi_1} |\psi(s,u(s-\iota(s,u(s))))|ds$$
$$\leq \int_0^T \phi(s,u'(s-\tau))ds$$
$$+ \int_{\Xi_2} |\psi[s,u\left(s-\iota(s,u(s))\right)]|ds$$
$$+ \int_{\Xi_3} |\psi[s,u\left(s-\iota(s,u(s))\right)]|ds.$$

Thus, by (3), we obtain

$$\| u \| \le \int_0^T |u''(s)|ds$$

$$\le \int_0^T |\phi(s,u'(s-\tau))|ds$$

$$+ \int_0^T |\psi(s,u(s-\iota(s,u(s))))|ds$$

$$+\overline{\beta}\int_0^T |D^{\wp}u(s)|ds + \int_0^T |\eta(s)|ds$$

$$\le \int_0^T |\phi(s,u'(s-\tau))|ds + [\int_{\Xi_1}$$

$$+\int_{\Xi_2} + \int_{\Xi_3}] |\psi[s,u\left(s-\iota\left(s,u(s)\right)\right)]|ds$$

$$+\overline{\beta}\int_0^T |\frac{d}{ds}\int_0^s \frac{(s-\tau)^{-\wp}}{\Gamma(1-\wp)}u(\tau)d\tau|ds$$

$$+\int_0^T |\eta(s)|ds \le 2\int_0^T |\phi(s,u'(s-\tau))|ds$$

$$+[\int_{\Xi_2} + \int_{\Xi_3}] |\psi[s,u\left(s-\iota\left(s,u(s)\right)\right)]|ds$$

$$+\overline{\beta}\int_0^s |\frac{d}{ds}\int_0^T \frac{(s-\tau)^{-\wp}}{\Gamma(1-\wp)}u(\tau)ds|d\tau + T\|\eta\|$$

$$\le 2T(\lambda_1\|u\|+\lambda_2) + 2T(\lambda_3\|u\|+\lambda_4)$$

$$+\frac{\overline{\beta}T^{1-\wp}}{\Gamma(1-\wp)}\|u\|.$$

Hence, we attain

$$\|u\| \le \frac{2T(\lambda_2+\lambda_4)}{1-(2T\lambda_1+2T\lambda_3+\frac{\overline{\beta}T^{1-\wp}}{\Gamma(1-\wp)})}.$$

This completes the proof.

The existence of Equation (1) can be explained by utilizing the upper and lower bound and can be converted as a periodic boundary-value problem by the formula :

$$u''(t) = F(t,u,D^{\wp}u), \qquad (6)$$

satisfying the boundary conditions

$$u(0)=u(T), u'(0)=u'(T), t\in J=[0,T]$$

where $F: J\times R\times R \to R$ is a Caratheodory function. Recall that the function is called a Caratheodory function if it is measurable on J continuous on R^2 and bounded by a Lebesgue integrable function ($|F(t,u,D^{\wp}u)| \le l(t)$, $l(t)\ge 0$). A function $\bar{u} \in B$ is called an upper solution of (6) if it achieves

$$\bar{u}''(t) \le F(t,u,D^{\wp}\bar{u}), u(0)=\bar{u}(t), \bar{u}'(0) \le \bar{u}'(T) \qquad (7)$$

and $\underline{u}$ is called a lower solution if

$$\underline{u}''(t) \ge F(t,\underline{u},D^{\wp}\underline{u}), \underline{u}(0)=\underline{u}(T), \underline{u}'(0) \ge u'(T) \qquad (8)$$

A function u is called a solution of (6) if it is an upper and a lower solution to (6). It is clear that $\underline{u}(t) \le \bar{u}(t), t\in J$.

Theorem 3.2 : Let Eq. (6) has an upper solution and a lower solution. Moreover, there exist constants $k_1 > 0$ and $k_2 > 2$. Hence,

$$\underline{u}(t) \le u_1(t) \le u_2(t), D^{\wp}u_1(t) \le D^{\wp}u_2(t),$$
$$F(t,u_2,D^{\wp}u_2) - F(t,u_1,D^{\wp}u_1)$$
$$\ge K_2(D^{\wp}u_2 - D^{\wp}u_1) - K_1(u_2-u_1). \qquad (9)$$

Then Equation (6) has a solution $u(t)$ satisfying

$$K_1(t) \le u(t) \le K_2(t),$$

where for some a constant $\kappa \in R$

$$K_1(t) := \underline{u}'(t) + \kappa\underline{u}(t), K_2(t) := \bar{u}'(t) + \kappa\bar{u}(t).$$

Proof : From (9), we obtain

$$F(t,u_2,D^{\wp}u_2) \ge F(t,u_1,D^{\wp}u_1), u_2 \ge u_1.$$

It is clear that

$$K_2'(t) + \kappa K_2(t) \le F(t, K_2(t), D^{\wp}K_2(t)), K_2(0) = K_2(T)$$

and

$$K_1'(t) + \kappa K_1(t) \ge F(t, K_1(t), D^{\wp}K_1(t)), K_1(0) = K_1(T).$$

Our aim is to derive that

$$u(t) \le K_2(t).$$

Let

$$\mu(t) := u(t) - K_2(t),$$

Then

$$\mu(0) \ge \mu(T).$$

If $\mu(t) > 0$, for some $t \in J$, then by the maximum-minimum principle, for $\kappa > 0$ the minimum of $\mu(t)$ is nonnegative and for $\kappa < 0$, the maximum of $\mu(t)$ is non-positive, we have

$$u'(t) + \kappa u(t) \ge K_2'(t) + \kappa K_2(t) \ge \mu'(t) + \kappa\mu(t) \ge 0.$$

Thus, it can be concluded that $\mu(t) \leq 0, t \in J$, which is a contradiction. Therefore, there would be a point $s \in J$ with

$$\mu(s) \leq 0 \Rightarrow u(s) \leq K_2(s).$$

Similarly, it can be expressed that $u(s) \geq K_1(s)$. Hence, we attain

$$K_1(s) \leq u(s) \leq K_2(s), s \in J.$$

This completes the proof.

This algorithm can be applied on a swarm of insects milling. Each insect selects a set of neighbors (living or inert) and transfer either towards or away from them. The ants pick the five closest termites; the termites select the nearest soil pellets; the ladybugs pick the two ants closest to the powdered sugar that is not in the vicinity of any termite; and so on. Once the collection is made, each agent appeals a second process, this time to move to a location determined entirely by the characteristics and locations of its neighbours. To model this kind of multi-agent dynamics, following formula was suggested :

$$u''_i(t) + \beta(u_i(t))D^{\varphi}u_i(t) + \phi(t, u_i'(t - \tau)) + \psi[t, u_i(t - \iota(t, u_i(t)))] = \eta(t). \quad (10)$$

It is a new model, but only new to the extent that it unifies an extensive variety of well-studied domain-dependent systems. The importance of any new system is to study its stability in terms of Hadamard well-posed. Since the system (10) is formulated for multi-agent users, it is enough to show that (10) has multi-solutions. In view of Theorem 3.1, it is conclude that the system (10) is stable. Consequently, the system is complete; hence the mildest perturbation creates periodic behavior. While ϕ and ψ can be considered as arbitrary functions, the attitude behind system (10) is to keep ϕ simple so that developing phenomena can be attributed not so much to the power of individual agents, but to the flow of information across the communication network $\psi(u)$. By characteristic ψ from ϕ, the model also separates the syntactic. The properties of natural algorithm would then be consequent algorithmically.

Moreover, system (10) models how people transfigured opinions over time as an effect of human communication and knowledge achievement. Instead of appreciating their technique toward information, though, individuals are destined to recycle the similar opinions in perpetuity, there is a deep theoretical understanding. The communication structure joins every agent to some of its nearest neighbors. Mathematically, the function ψ requires knowledge about everyone, while the function ϕ describes each agent to support its velocity with those of its neighbors. Some creatures can sense food gradients and direct their signal accordingly. In case of bacterial chemotaxis, the stimuli are so weak that the creatures are concentrated to act in a random walk with a drift toward higher food attentions. System (10) can model these processes with the use of both motile and inert agents. Chemotaxis is typically preserved as an asocial process (single agents interacting only with the environment). It has been detected that training behavior can simplify gradient climbing for creatures, a case where living in groups enhances searching ability.

In a conclusion, the natural differential equations are generalized, by utilizing the concept of the fractional calculus. Some sufficient conditions have been illustrated for the existence of periodic outcomes to a fractional Rayleigh-type equation with state- dependent delay. There were two different methods for the findings; the first one was applied, by using inequality theory, while the second one was employed, by defining the upper and lower solutions.

Acknowledgment

The authors would like to thank the referees for giving useful suggestions that improved the quality of this paper.

References

Abbas, S., M. Benchohra, J. Henderson and B. A. Slimani: Existence and attractivity for the Darboux problem of fractional order neutral differential equations. *J. App. Mathem. Comp.*, **15**, 1-13 (2015).

Alzabut, J. and C. Tunc: Existence of periodic solutions for a type of Rayleigh equation with state-dependent delay. Electron. *J. Diff. Equ.*, **77**, 1-8 (2012).

Anguraj, A., M. C. Ranjini, M. Rivero, and J. J. Trujillo: Existence results for fractional neutral functional differential equations with random impulses. *Mathematics*, **3**, 16-28 (2015).

Deimling, K. : Nonlinear Functional Analysis, Springer, Berlin (1985).

Durrett, R. : *Probability models for DNA sequence evolution*. Springer Science & Business Media, 2008.

Gains, R.E. and J.L. Mawhin : Coincidence Degree and Nonlinear Differential Equations. Lecture Notes in Mathematics, 568, Springer, Berlin (1977).

Ibrahim, R. W., M. Z. Ahmad, and H. F. Al-Janaby: Mathematical model for adaptive evolution of populations based on a complex domain. *Saudi J. Biol. Sci.*, **23.1**, S45-S49 (2016).

Ibrahim, R.W. and J.M. Jahangiri: Boundary fractional differential equation in a complex domain. *Bound. Val. Probl.*, **66**, 1-13 (2014).

Ibrahim, R.W. and J.M. Jahangiri: Existence and uniqueness of an attractive nonlinear diffusion system. *Applied Mathematics and Computation*, **257**, 169-177 (2015).

Ren, J., Z. Cheng, and S. Siegmund: Neutral operator and neutral differential equation. *Abstr. Appl. Anal.*, ID 969276 (2011).

Xin, Y. and S. Zhao: Existence of periodic solution for generalized neutral Rayleigh equation with variable parameter. *Adva. Diff. Equ.*, **209**, 1-14 (2015).